Oscillation frequencies of tapered plant stems

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Free oscillations of upright plant stems, or in technical terms, slender tapered rods with one end free, can be described by considering the equilibrium between bending moments in the form of a differential equation with appropriate boundary conditions. For stems with apical loads, where the mass of the stem is negligible, Mathematica 4.0 returns solutions for tapering modes \( \alpha = 0, 0.5, \) and 1. For other values of \( \alpha \), including cases where the modulus of elasticity varies over the length of the stem, approximations leading to an upper and a lower estimate of the frequency of oscillation can be derived. For the limiting case of \( \omega = 0 \), the differential equation is identical with Greenhill’s equation for the stability against Euler buckling of a top-loaded slender pole. For stems without top loads, Mathematica 4.0 returns solutions only for two limiting cases, zero gravity (realized approximately for oscillations in a horizontal orientation of the stem) and for \( \omega = 0 \) (Greenhill’s equation). Approximations can be derived for all other cases. As an example, the oscillation of an \( Arundo donax \) plant stem is described.

Key words: \( Arundo donax \); differential equations; free oscillation; taper; top load.

Knowledge of the resonance frequency particularly in the fundamental mode of oscillation is necessary to assess a plant’s stability against dynamic wind loads (Kerzenmacher and Gardiner, 1998). In addition measurement of the frequency of oscillations of plant stems or slender rods offers an efficient and potentially nondestructive way to determine mechanical properties such as the storage modulus and the loss modulus of elasticity (Young, 1989; Vincent, 1990; Zebrowski, 1991; Niklas, 1992; Speck and Spatz, 2000). Niklas and Moon (1988) and Niklas (1989, 1990, 1991, 1997) have recorded multiple resonance spectra and evaluated the modulus of elasticity for a variety of plant stems. Zebrowski (1999) and Spatz and Zebrowski (2001) measured the natural frequency of free vibrations of upright cereal plant shoots with an apical load. They described oscillations in the vertical and in the horizontal orientation of the stems by an equilibrium of bending moments, being different for the two orientations. Solutions of the corresponding differential equations were presented for untapered columns with negligible self mass.

The present approach generalizes this to plant stems with or without apical loads, with different tapering modes and with changes of the modulus of elasticity and of density from the base to the apex as usually observed in plant stems. Geometric input parameters are the total length, the cross-sectional area and the second moment of area of the stem at the base. The results, therefore, relate to any cross section if, at least for cross sections with a symmetry less than threefold, the plane of bending is specified.

Several examples of the solution of the differential equation are presented and approximations for others are outlined. Intermediate values can be interpolated or calculated directly by the Mathematica 4.0 program given. The accuracy of the approach is tested against the oscillation frequency of a tapered plastic rod. Its applicability for plant stems is exemplified by quantitative analyses of video recordings of an \( Arundo donax \) stem.

MATERIALS AND METHODS

\( Arundo donax \) stems in a wind-sheltered area of the Botanical Garden, Freiburg, Germany, were bent and released by hand. The oscillations were recorded in side view with a video camera. A canvas of \( 3 \times 4 \) m was used as background for the videos and provided additional wind shelter. The stems carried small markers of high contrast tape at eight positions along the stem. Their displacement from the resting position was analyzed frame by frame with the software SIMI Motion 4.6 (SIMI GmbH, Unterschleissheim, Germany).

COMPUTATIONS

Calculations were carried out with the powerful program Mathematica 4.0 (Wolfram Research, Champaign, Illinois, USA). Values for \( A, B, G, \) and \( H \) (Eq. 19) are found by nesting intervals. Finding the least positive singularities requires small distortions of the boundary conditions as described by Spatz (2000). All computations were carried out to at least five significant digits. Mathematica 4.0 does not solve the integrals (Eqs. 12, 16, and 19) directly; instead, a table of values is provided and interpolated by polynomials. These can be integrated.

THEORETICAL CONSIDERATIONS

Top load, negligible mass of the stem—Vertical orientation of the stem—The differential equation for free vibrations of an upright slender rod or stem with an apical load (Fig. 1) can be formulated as an equilibrium of bending moments. Neglecting the mass of the stem itself we obtain for deflections within the linear elastic range

\[
EI\frac{d^2y}{dx^2} + Mg(y - y_T) - (\omega^2 + \delta^2)M(x - x_T)y_T = 0 \quad (1)
\]

with \( x = \) coordinate along the length of the stem \( (x = L \) at the base and \( x = 0 \) at the tip of the stem); \( x_T = \) position of the top load; \( y = \) deflection from the resting position; \( y_T = \) deflection at \( x = x_T; \) \( L = \) length of the stem; \( E = \) modulus of elasticity; \( I = \) axial second moment of area; \( M = \) mass of the apical load; \( g = \) gravitational acceleration; \( \omega = \) angular frequency; and \( \delta = \) decay constant.

For simplicity we will treat only slightly damped oscillations so that

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For plant stems it is reasonable to assume that \( EI \) varies smoothly and monotonously over its length. We can approximate

\[
EI = E_B I_B z^{a_1 b}
\]

(2)

where \( z = x/L \) and correspondingly \( dx = Ldz \); \( E_B \) = modulus of elasticity at the base; \( I_B \) = axial second moment of area at the base; \( a = \) tapering mode \( (r = r_B z^a \) or \( I = I_B z^a) \); and \( b = \) mode of dependence of the modulus of elasticity \( (E = E_B z^b) \).

With \( \psi = y/y_T \), Differential Eq. 1 can be rewritten in the form

\[
\frac{d}{dz} \left( z^{a_1 b} \frac{d^2 \psi}{dz^2} \right) + B \frac{d\psi}{dz} - A = 0 \tag{3a}
\]

with

\[
B = L^2 \frac{Mg}{E_B I_B} \quad \text{(the "gravitational term")} \tag{3b}
\]

\[
A = L^3 \frac{M}{E_B I_B} \omega^2 \quad \text{(the "acceleration term")} \tag{3c}
\]

The boundary conditions read

\[
\psi[1] = 0,
\]

\[
\frac{d\psi}{dz}(0) = 0 \quad \text{for the base} \quad \text{and} \tag{3d}
\]

\[
\frac{d^2 \psi}{dz^2}(z_T) = 0 \quad \text{for the relative position of the top load}. \tag{3e}
\]

For \( A = 0 \) (i.e., \( \omega = 0 \)) Eq. 3a–e becomes identical to Greenhill’s equation for the stability of a top-loaded slender rod against Euler buckling (Greenhill, 1881; Spatz, 2000).

The remaining problem is to find numerical values for combinations of \( A \) and \( B \), which characterize the fundamental frequency of free vibration. The equivalent condition for any given value of \( B \) is the solution of Differential Eq. 3a with the boundary conditions (Eq. 3d), which fulfills \( \psi(z_T) = 1 \).

Figure 2 is an example of a solution of Differential Eq. 3 as returned by Mathematica 4.0. The output gives the function \( \psi(z) \). The plot \( \psi(z) \) against \( z \) displays the actual bending line.

Figure 3 shows how the acceleration term \( A \) and therefore \( \omega \) according to Eq. 3c depends on the gravitational term \( B \) for those values of the tapering mode for which a solution is available. The lines through the data points are fitted by a second-order polynomial

\[
A = -0.0065 B^2 - 1.262 B + 3.4988 \quad \text{for} \quad \alpha = 0;
\]

\[
A = -0.2527 B^2 - 1.643 B + 1.4307 \quad \text{for} \quad \alpha = 0.5;
\]

and

\[
A = -94.102 B^2 - 4.6722 B + 0.17387 \quad \text{for} \quad \alpha = 1.
\]

The frequency of oscillation depends strongly on the position of the top load. Figure 4 gives an example for \( \alpha = 0.5 \). It is physically impossible to apply a weight on the very tip of a tapering column. Correspondingly for \( \alpha \) or \( B \neq 0 \) solutions are only obtained for \( z_T > 0 \), i.e., for a truncated column.

**Horizontal orientation of the stem**—Another situation is encountered if the stem is oriented horizontally. Neglecting the mass of the stem, the differential equation describing the equilibrium of bending moments reads

\[
EI \frac{d^2 y}{dx^2} + Mg(x - x_T) - (\omega^2 + \beta^2) M(x - x_T) y_T = 0. \tag{4}
\]

If we set \( z = x/L \) and \( \eta = \frac{1}{y_T} \times \frac{y - g/\omega_0}{y_T} \) we obtain for \((\omega^2 + \beta^2) = \omega_0^2\)

\[
EI \frac{d^2 \eta}{dz^2} - L^3 M \omega_0^2 (z - z_T) = 0 \tag{5}
\]

with Eq. 2 Differential Eq. 5 can be written in the form

\[
\frac{d}{dz} \left( z^{a_1 b} \frac{d^2 \eta}{dz^2} \right) - A = 0 \quad \text{(6a)}
\]

with

\[
A = L^3 \frac{M}{E_B I_B} \omega_0^2. \tag{6b}
\]

The boundary conditions read

\[
\psi[1] = 0,
\]

\[
\frac{d\psi}{dz}(0) = 0 \quad \text{for the base and} \tag{6d}
\]

\[
\frac{d^2 \psi}{dz^2}(z_T) = 0 \quad \text{for the relative position of the top load}. \tag{6e}
\]
Fig. 2. Formulation of the Differential Eq. 3a–e and its solution by Mathematica 4.0. This is given in form of a complex function. The imaginary part is of the order of $10^{-15}$. The graph displays the actual bending line, with $z = 1$ at the base and 0 at the very tip of the stem. The correct value for $A$ is found as the solution that fulfills $\psi_0 = 1$. The correct value for $B$ is found as singularity of the differential equation. The singularity lies in the interval where the solution switches from positive to negative values.

\[ f[z_] = \\
\psi[z] /. \\
Flatten[
\text{DSolve}[[\partial_z \left( z^{4 \alpha \beta} \psi''[z] \right) - A + B \psi'[z] = 0, \\
\psi[1] = 0, \psi'[1] = 0, \psi''[zT] = 0.001}, \\
\psi[z], z]]
\]

\[ f[zT] \\
\text{Plot}[f[z], \{z, \text{Tip}, 1\}]
\]

\[ 0.908384 - 2.10942 \times 10^{-15} \ I - \\
(1.37045 + 0.59653 \ I) z^{0.5 - 0.387298 \ I} - \\
(1.37045 - 0.59653 \ I) z^{0.5 + 0.387298 \ I} + 1.83253
\]

\[ 1. - 1.27676 \times 10^{-15} \ I
\]

\[ \eta(1) = 0, \]
\[ \frac{d\eta}{dz}(1) = 0 \quad \text{for the base and} \quad (6c) \]
\[ \frac{d^2\eta}{dz^2}(z_T) = 0 \quad \text{for the relative position of the top load} \quad (6d) \]

Eq. 6a–d is equivalent to Eq. 3a–e for $B = 0$, i.e., in principle for zero gravity. It should be noted though that Eq. 4 is only valid if the stem in its resting position can be considered nearly horizontal over its entire length. This requires a very stiff rod, such that the downward deflection is small. From cantilever theory it follows that this can be expressed by
Fig. 3. Solutions of Differential Eq. 3a±e. The “acceleration term” $A$ is plotted against the “gravitational term” $B$ for tapering modes $\alpha = 0, 0.5, 1$, $\beta = 0$, and $z_t = 0.05$. The lines through the data points are fitted by second order polynomials (see text). The data for $\alpha = 1$ are plotted on a different scale in Fig. 9.

Fig. 4. The “acceleration term” $A$ plotted against the “gravitational term” $B$ for $\alpha = 0.5$, $\beta = 0$, and different positions $z_t$ of the top load. The data points are fitted by second-order polynomials.

Stems without top load—Vertical orientation of the stem—
The differential equation for free vibrations of an upright slender rod or stem without an apical load (Fig. 6) can be formulated as an equilibrium of line loads.

$$\frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right) + \frac{d}{dx} \left[ \int_0^x \rho a \frac{dy}{dx} \right] - \omega^2 \rho ay = 0 \quad (7)$$

where $a$ is the cross-sectional area with $a = a_u z^2$, $a_u$ the cross-sectional area at the base, and $\rho$ the density of the stem at a particular height. For plant stems the variation of $\rho$ along the stem can be approximated by $\rho = \rho_u z^\gamma$ with $z = x/L$, where $\rho_u$ is the density at the base and $\gamma$ the mode of dependence.

Eq. 7 can be rewritten in the form

$$\frac{d^2}{dz^2} \left( \frac{E_l}{a} \frac{d^2y}{dz^2} \right) + G \frac{d}{dz} \left( \frac{z^{2\alpha + \gamma + 1} \frac{dy}{dz}}{2\alpha + \gamma + 1} \right) - H z^{2\alpha + \gamma} \frac{dy}{dz} = 0 \quad (8a)$$

with

$$G = \frac{L^3 \rho_u a_u \omega}{E_l I_b} \quad \text{(the “gravitational term”)} \quad \text{and (8b)}$$

$$H = \frac{L^3 \rho_u a_u \omega^2}{E_l I_b} \quad \text{(the “acceleration term”)} \quad \text{(8c)}$$

The boundary conditions read

$$y(1) = 0 \quad \frac{dy}{dz}(1) = 0 \quad \text{for the base} \quad \text{and (8d)}$$

$$\frac{d^2y}{dz^2}(0) = 0 \quad \frac{d^2y}{dz^2}(0) = 0 \quad \text{for the tip} \quad \text{(8e)}$$

For $H = 0$ (i.e., $\omega = 0$) Eq. 8a±e becomes identical to Greenhill’s equation for the stability of a slender pole without top load (Greenhill, 1881; Spatz, 2000).

The remaining problem is to find numerical values for $H$ and $G$, which characterize the fundamental frequency of free vibration. The equivalent condition is the least positive sin-
Fig. 5. Solutions of Differential Eq. 6a–d for oscillations with horizontal orientation of the stem are available for almost all combinations of $\alpha$ and $\beta$. The lines through the data points are fitted by third-order polynomials.

Fig. 6. Scheme illustrating a slender tapered plant stem without a top load oscillating around an upright resting position.

Regular oscillation in the solution of the differential equation with the above boundary conditions. Mathematica 4.0 solves Differential Eq. 8a–e only for $G = 0$ or $H = 0$ (Fig. 7). Approximations for $G$ and $H \neq 0$ can be obtained as outlined below.

**Horizontal orientation of the stem**—Exact solutions of Differential Eq. 8a–e are obtained if the gravitational term $G = 0$, i.e., for oscillation in the horizontal direction, provided that the downward deflection in the resting position is small. Figure 8 shows the acceleration term $H_0$ and therefore $\omega$ according to Eq. 8c as a function of the tapering mode $\alpha$, the mode of dependence of the modulus of elasticity $\beta$, and the mode of dependence of the gravity $g$ along the stem as typically observed for plant stems (V. Fässler and H.-C. Spatz, unpublished data). The data can be approximated by fifth-order polynomials (Table 1). For this and all other cases considered intermediate values can be interpolated or computed directly with Mathematica 4.0 as shown in Figs. 2 and 7.

**APPROXIMATIONS**

**Top load, negligible mass of the stem**—Differential Eq. 3a–e is only solved for $4\alpha + \beta = 0$, 2 or 4 with independent $\alpha$ and $\beta$. For all other solutions, equations are only provided for $A = 0$ or $B = 0$. As noted before, for $A = 0$ Differential Eq. 3a–e is identical to Greenhill’s equation, and $B = 0$ corresponds to zero gravity or is realized by a stem in the horizontal position being nearly straight (see above).

For all other cases approximations can be obtained by considering the potential energy during oscillations (Timoshenko and Young, 1948).

The strain energy of bending the stem is

$$ V_1 = \frac{1}{2} \int_{\zeta_T}^L \text{Moment Curvature} \; dx. $$

Within the linear elastic range where Moment = $EI$ Curvature and with $\zeta = x/L$ and Eq. 2 this reads

$$ V_1 = \frac{1}{2} \frac{E_b I_b}{L^3} \int_{\zeta_T}^1 \left( \frac{d^2 y}{dz^2} \right)^2 \; dz. \quad (9) $$

During the oscillation the top load experiences a vertical displacement, $\Delta h$. The corresponding potential energy is

$$ V_2 = -Mg\Delta h = -\frac{1}{2} \frac{Mg}{L} \int_{\zeta_T}^1 \left( \frac{dy}{dz} \right)^2 \; dz. \quad (10) $$

Because the oscillation converts the potential energy to kinetic energy and vice versa, the budget reads

$$ V = V_1 + V_2 = T = \frac{1}{2} M \omega^2 y_T. \quad (11) $$

With $\psi = y/\gamma_T$, Eq. 3a and 3b and the abbreviations
Fig. 7. Formulation of Differential Eq. 8a–e and its solution by Mathematica 4.0 in terms of hypergeometric functions. The correct values for $G$ with $H = 0$ and $G$ with $H = 33.1786$ are found as singularities of the differential equation. As in Fig. 2, the graph displays the actual bending line, with $z = 1$ at the base and $0$ at the tip of the stem.

$$f[z_] = y[z] / . Flatten[
\text{DSolve}\left[\left\{\frac{\partial}{\partial z} \left( z^{\alpha + \beta} y''[z] \right) - H z^{2\alpha + \gamma} y[z] + \frac{G}{(2 \alpha + \gamma + 1)} \frac{\partial}{\partial z} \left( z^{2\alpha + \gamma + 1} y'[z] \right) = 0, \right\}
\right.
\left\{y[1] = 0, y'[1] = 0, y''[\text{Tip}] = 0, \right\}
\right]\]
$$

Plot[$f[z], \{z, \text{Tip}, 1\}$]

```
Out[162]= 0.0136738 HypergeometricPFQ[{}, {0.5625, 0.6875, 0.8750}, 0.316416 z^{16/5}],
         1.94873 
\times 10^{-9} z^{2/5} HypergeometricPFQ[{}, {0.6875, 0.8125, 1.125}, 0.316416 z^{16/5}],
         0.0217411 z HypergeometricPFQ[{}, {0.8750, 1.1875, 1.3125}, 0.316416 z^{16/5}],
         1.45699 \times 10^{-6} z^{7/5} HypergeometricPFQ[{}, {1.1250, 1.3125, 1.4375}, 0.316416 z^{16/5}]
```

Fig. 7. Formulation of Differential Eq. 8a–e and its solution by Mathematica 4.0 in terms of hypergeometric functions. The correct values for $G$ with $H = 0$ and $H = 33.1786$ are found as singularities of the differential equation. As in Fig. 2, the graph displays the actual bending line, with $z = 1$ at the base and $0$ at the tip of the stem.

$$K_1 = \int_{\gamma}^{1} z^{\alpha + \beta} \left( \frac{d^2 \psi}{dz^2} \right)^2 \, dz \quad \text{and} \quad K_2 = \int_{\gamma}^{1} \left( \frac{d \psi}{dz} \right)^2 \, dz$$

lead to

$$A = K_1 - K_2 B = A_0 - K_2 B. \quad (12)$$

The evaluation of the numerical terms $K_1$ and $K_2$ requires the knowledge of the bending line $y(z)$. Timoshenko and Young (1948) used a cosine function. A better approximation, especially for strongly tapered columns, uses the bending lines for the two limiting cases: (1) For $A = 0$, $\psi(z)$ is chosen as solution of Eq. 3a–e with $A = 0$, i.e., Greenhill’s equation for
Fig. 8. Solutions of Differential Eq. 8a–e for \( G = 0 \), i.e., for oscillations with horizontal orientation of the stem, are available for almost all combinations of \( \alpha, \beta, \) and \( \gamma \). The lines through the data points are fitted by fifth-order polynomials (Table 1).

Euler buckling; (2) For \( B \approx 0 \), \( \psi(z) \) is chosen as solution of Differential Eq. 3a–e with \( B = 0 \), i.e., zero gravity or horizontal orientation provided that the downward deflection in its resting position is small.

This is illustrated for \( \alpha = 1 \) in Fig. 9. Applying Eq. 10 and choosing a solution of Eq. 3a–e for \( A = 0 \) leads to a straight line through \((0, A_0)\) and \((B^*, 0)\), and choosing a solution of equation Eq. 3a–e for \( B = 0 \) leads to a straight line through \((0, A^*)\) and \((B_0, 0)\), where \( A_0 = A(B = 0) \) and \( B_0 = B(A = 0) \) are exact solutions.

\[ A^* = K_1 \quad \text{for} \quad B = 0, \quad B^* = K_1/K_2 \quad \text{for} \quad A = 0. \]

Table 1. Within deviations of <0.5% the data shown in Fig. 8 can be approximated by \( H_0 = c_0 + c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 + c_4 \alpha^4 + c_5 \alpha^5 \). The table lists values for \( c_0 \) to \( c_5 \) for some values of \( \beta \) and \( \gamma \) typically found in plant stems.

<table>
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<th>( \beta )</th>
<th>( \gamma )</th>
<th>( c_0 )</th>
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</tr>
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The lines from \((0, A_0)\) or \((B_0, 0)\) to the intercept of the two straight lines represent an upper estimate of the actual data points, a straight line through \((0, A_0)\) and \((B_0, 0)\) a lower estimate. Table 2 lists values of \( A_0, A^*, B_0, \) and \( B^* \) for tapering modes \( \alpha \) between 0 and 1.5.

In plant stems the modulus of elasticity usually varies from bottom to top. This can approximately be taken into account by an appropriate choice of the mode of dependence of the
modulus of elasticity. Figure 10 gives the values of \( A_o, A^*, B_o, \) and \( B^* \) as function of \( \beta \) for a tapering mode \( \alpha = 0.25 \).

**Stems without top load**—Differential Eq. 8a–e is only solved by Mathematica 4.0 for \( G = 0 \) (zero gravity) or \( H = 0 \) (Greenhill’s equation). For all other cases approximations must be obtained by considering the energy budget during oscillation (Timoshenko and Young, 1948).

The strain energy of bending the stem can be calculated analogically to Eq. 9

\[
V_1 = \frac{1}{2} \frac{E_b I_B}{L^3} \int_0^1 z^{2a+\gamma} \left( \frac{d^2 y}{dz^2} \right)^2 dz.
\]  

During the oscillation each part of the stem experiences a vertical displacement. The corresponding potential energy is

\[
V_2 = -\frac{1}{2} \rho_b a_B \int_0^1 z^{2a+\gamma} \left[ \frac{1}{1} \left( \frac{d y}{d \xi} \right)^2 d \xi \right] dz.
\]  

The kinetic energy can be calculated as

\[
T = \frac{1}{2} \rho_b a_B \omega^2 \int_0^1 z^{2a+\gamma} y^2 dz.
\]  

The energy balance \( V = V_1 + V_2 = T \) with Eq. 8b–c and the abbreviations

\[
M_1 = \int_0^1 z^{2a+\gamma} \left( \frac{d^2 y}{dz^2} \right)^2 dz,
\]

\[
M_2 = \int_0^1 \left[ \frac{1}{1} \left( \frac{d y}{d \xi} \right)^2 d \xi \right] dz \]

and

\[
M_4 = \int_0^1 z^{2a+\gamma} y^2 dz
\]

leads to

\[
H = \frac{M_1}{M_4} - \frac{M_1}{M_4} \frac{G}{M_4} = \frac{H_o - M_1}{M_4} G.
\]  

The bending line \( y(z) \) for \( H = 0 \) is approximated as the solution of Differential Eq. 8a–e for \( H = 0 \), i.e., Greenhill’s equation. The bending line \( y(z) \) for \( \beta \neq 0 \) is approximated as the solution of Differential Eq. 8a–e for \( B = 0 \), i.e., zero gravity.

Table 3 gives values for \( G_o, G^*, H_o, \) and \( H^* \) as function of the tapering mode \( \alpha \).

\[
G^* = M_1/M_4 \quad \text{for} \quad H = 0, \quad \text{and} \quad H^* = M_1/M_4 \quad \text{for} \quad G = 0.
\]  

By analogy we infer that the exact solutions lie between a straight line through \( (G_o, 0) \) and \( (0, H_o) \) and straight lines from \( (G_o, 0) \) and \( (0, H_o) \) to the intercept of the lines from \( (G_o, 0) \) to \( (0, H^*) \) and \( (G^*, 0) \) to \( (0, H_o) \) (compare Fig. 9). For \( \alpha \leq 1 \), the differences between \( G_o \) and \( G^* \) and \( H_o \) and \( H^* \) are small, such that the straight line from \( (G_o, 0) \) to \( (0, H_o) \) serves as a good approximation (Fig. 11). Values for \( \beta \neq 0 \) and/or \( \gamma \neq 0 \) are not tabulated for this case. They can easily be computed using the Mathematica 4.0 program shown in Fig. 7.

**Generalization**—The energy balance (Timoshenko and Young, 1948) can be extended to generalize the approach for upright tapered stems with nonnegligible mass and an additional load (mass) attached at one point along the stem. The strain energy of bending the stem can be calculated according to Eq. 13.

During the oscillation each part of the stem as well as the additional load experiences a vertical displacement.
From this the oscillation frequency can be obtained as in Eqs. 3b, 3c, 8b, and 8c and with

\[ \omega = \left( \frac{N_1 - \frac{L}{E_n I_n} (MN_2 + \rho_n a_n L N_3)}{L^2 \rho_n a_n L} \right)^{1/2} \]

As outlined above the evaluation of the numerical terms \( N_1, N_2, N_3, \) and \( N_4 \) requires the knowledge of the normalized bending function \( \psi(z) \). In general this is only available if three of the terms \( A, B, G, \) or \( H \) are zero, while the fourth is determined as a solution of Differential Eqs. 3a–e or 8a–e. The numerical terms \( N_1, N_2, N_3, \) and \( N_4 \) can be computed for the corresponding function \( \psi(z) \) being different for the four limiting cases \( A_0, B_0, G_0, \) or \( H_0 \). Table 4 shows the results of these computations for a given set of input values \( \alpha, \beta, \gamma, z_T, \) and \( z \). The values for \( A^*, B^*, G^*, \) and \( H^* \) allow to construct plots as in Fig. 9 to determine upper and lower estimates for the oscillation frequencies or for the stability against Euler buckling. Some special cases deserve mentioning. (a) \( G = 0; H = 0 \) describes the case of a stem with a top load but negligible mass of the stem itself as discussed above; (b) \( A = 0 \) and \( B = 0 \) describes the case of a stem with nonnegligible mass and no top load as discussed above; (c) \( B = 0; G = 0 \) describes the oscillation of a stem in the horizontal orientation with non-negligible mass and an additional mass attached (Spatz and Zebrowski, 2001). For \( \alpha = 0, \beta = 0, \gamma = 0, \) and \( z_T = 0 \) and the oscillation frequency being dominated by the additional mass \( M \) the upper estimate reads

\[ \omega^* = 3.00EI/(ML^3 + 0.236\rho a L^2) \]
Table 4. Values for the approximation of oscillation frequencies for $\alpha = 0.5$, $\beta = 0$, $\gamma = 0$, $\varepsilon_T = 0.05$, and $z(\text{tip}) = 0.0009$.

<table>
<thead>
<tr>
<th>$A_0$</th>
<th>$A = 0$</th>
<th>$B = 0$</th>
<th>$G = 0$</th>
<th>$H = 0$</th>
<th>$A^*$</th>
<th>$B^*$</th>
<th>$G^*$</th>
<th>$H^*$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$N_4$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4328</td>
<td>1.6188</td>
<td>1.7175</td>
<td>1.7625</td>
<td>0.8533</td>
<td>0.077668</td>
<td>1.2452</td>
<td>1.3020</td>
<td>22.878</td>
<td>75.744</td>
<td>39.961</td>
<td>0.062630</td>
<td>0.049390</td>
<td>0.087010</td>
<td>0.088726</td>
</tr>
</tbody>
</table>

identical with the approximation given by Young (1989; p. 715). If the mass of the stem is dominating we obtain an upper estimate

$$\omega^2 = 3.09EI/(ML^3 + 0.250paL^4)$$

The lower estimate is given by

$$\omega^2 = 3.00EI/(ML^3 + 0.242paL^4);$$

(d) $A = 0; H = 0$ corresponds to $\omega = 0$, it therefore describes the stability against Euler buckling for a stem with a nonnegligible self mass and an additional load attached (Spatz, 2000).

(e) and (f) The remaining cases $A = 0; G = 0$ and $B = 0; H = 0$ are not easily realized.

The general case equation (Eq. 20) has to be solved for the four sets of values $N_1$ to $N_4$ given in Table 4. Thus we obtain four values $\omega^2 (A_0)$, $\omega^2 (B_0)$, $\omega^2 (G_0)$, $\omega^2 (H_0)$. The upper estimate for the oscillation frequency is the minimum of these four values. The lower estimate can be computed as

$$\omega^2 = \frac{A_0 - gL^2}{E_B I_T} \left( \frac{A_0}{B_0} + \rho_B a_B L \frac{A_0}{G_0} \right) \left( M + \rho_B a_B L \frac{A_0}{H_0} \right)$$

RESULTS AND DISCUSSION

As a test for the accuracy of the calculations the oscillation frequencies in the upright orientation of a polyvinyl chloride rod manufactured to give a tapering mode of $\alpha = 0.1415$ and a nontapered rod of the same material were recorded. Differential Eq. 8a–e was solved for $\beta = 0$, $\gamma = 0$, and $G = 0$ and the solution corrected for the reduction of the oscillation frequency due to a finite gravitational term according to Eq. 16. This results in a determination of the modulus of elasticity of 3.40 GPa for the tapered rod and 3.27 GPa for the nontapered rod compared to 3.32 GPa from independent three-point-bending experiments.

As an example of the applicability of the approach to the oscillation of plant stems the results of the analysis of video recordings of an *Arundo donax* stem are presented in Fig. 12. In this particular example the leaves were removed. The length of the stem is $L = 4.33$ m. The sample is part of a study of damping of oscillations performed on eight plants under various conditions (to be published elsewhere). The data are compatible with a damped harmonic oscillation, implying that the oscillation did not go beyond the linear elastic range of the material. The frequency of oscillation was $\omega = 3.648 \pm 0.004$ sec$^{-1}$. The analysis of the amplitude as function of the height along the plant stem shows that it is a bending oscillation. It is not compatible with a pendulum with a hinge in the transition zone from stem to underground rhizome.
The difficulty in the approach lies in the determination of the best fit for the values of $\alpha$, $\beta$, and $\gamma$. With *Arundo donax* as a hollow stem, the outer and the inner radius have to be known to determine the cross-sectional area and the second moment of area. A double logarithmic plot of the second moment of area has a slope of 0.79. This yields an effective $\alpha = 0.20$; $4\alpha + \beta$ is determined from measurements of the bending stiffness as function of the position along the stem. A larger data set (Spatz et al., 1997) gives a value of $4\alpha + \beta = 1.71 \pm 0.11$. This leads to $\beta = 0.93$, consistent with the shape of the bending line. The density was found independent of the position along the stem, i.e., $\gamma = 0$. The solution of Differential Eq. 8a–e for these values and $G = 0$ gives $H_0 = 16.39$. The correlation coefficient between the measured and the calculated amplitudes as function of the height above ground is $R^2 = 0.999$.

Eq. 16 written in the form

$$ \frac{H_0}{H} = 1 + \frac{M_G}{M_\ell} = 1 + \frac{M_\ell g}{M_\ell Lo^2} $$

leads to an approximate value $H = 12.56$. Inserting the data for the cross-sectional area, the second moment of area, and the density at the base, a value for the modulus of elasticity at the base of $E_0 = 5.21$ GPa is obtained. A set of six *Arundo donax* stems where oscillations were recorded under the same conditions yielded $E_0 = 4.83 \pm 0.76$ GPa as compared to 5.23 $\pm 1.25$ GPa determined by three-point bending on an independent set (H. Beismann, unpublished data). The example shows the applicability of the approach for plant stems. It should be noted though that the analysis actually refers only to a straight vertical or horizontal position of the rod. However, for a plant stem like *Arundo donax* the difference between the oscillation frequencies in these two orientations amounts to 30%. A deviation from a straight vertical orientation will lead to errors considerably smaller than that. The same argument holds true for cereal plant shoots with the ear as an apical load (Spatz and Zebrowski, 2001). Detailed reports of data on oscillation of plant stems (Arundo donax, Cyperus alternifolius, Equisetum hyemale, and Juncus effusus) are in preparation.

**LITERATURE CITED**

Greenhill, G. 1881. Determination of the greatest height consistent with stability that a vertical pole or mast can be made, and the greatest height to which a tree of given proportions can grow. *Proceedings of the Cambridge Philosophical Society* 4: 65–73.


